JOURNAL OF COMBINATORIAL THEORY (B) 14, 256–258 (1973)

A Note on Graphs with Prescribed Clique and Point-Partition Numbers

J. M. S. SIMÕES-PEREIRA

Secção de Matemática, Universidade de Coimbra, Coimbra, Portugal

Communicated by W. T. Tutte

Received October 10, 1972

The aim of this note is to prove an analog to Mitchem's generalization [1] of the well-known Zykov theorem [2] on the existence of graphs with prescribed clique and chromatic numbers, where we adopt for the point-partition numbers, instead of the definition given by Chartrand, Geller, and Hedetniemi [3], that given by Lick and White [4]. The present result includes Mitchem's theorems 1 and 2 and its proof is much shorter; in fact it seems to me, and this note illustrates this point of view, that the definition in [4] is more powerful than that in [3] but has not yet received the attention it deserves.

We recall that Lick and White [4] define an n-degenerate graph as a graph such that the minimum degree of each induced subgraph is at most n. The point-partition number $\rho_n(G)$ of a graph G is then the minimum number of subsets into which the vertex set $V(G)$ may be partitioned so that each subset induces an n-degenerate graph. If we denote the clique number of G, i.e., the maximum number of vertices in any complete subgraph of G, by $w(G)$, then we state:

THEOREM. *For any integers n, R, and d such that $n \geqslant 0$, $d \geqslant 2$, and $R \geqslant \{d/(n+1)\}$, a graph G exists with $\rho_n(G) = R$ and $w(G) = d$.*

Proof. The theorem is a generalization of Zykov's theorem which corresponds to the case $n = 0$; this case has been proved in [2], hence we take for granted that there exist graphs with no triangles and chromatic number R. Let G' be one of these graphs. Let H be the graph with nR vertices and no lines and form the composition $G'' = G'[H]$ (see Harary [5]).

Now $\rho_n(G'') \leqslant R$. In fact, G' having chromatic number equal to R, the vertex set $V(G')$ may be partitioned into R disjoint subsets of independent vertices. Consider all the vertices of G'' which may be associated, in an obvious way, by the definition of G'', to the vertices of G' in each one of these R subsets. They are also independent vertices in G'', hence the subgraph of G'' induced by them is totally disconnected, i.e., 0-degenerate and thus n-degenerate. The number of these (disjoint) subgraphs is also R. It remains to show that $\rho_n(G'')$ cannot be less than R. To this purpose, consider any partition of $V(G'')$ into less than R subsets. Denote them by $U_1, \ldots, U_{R-S}$ (with $1 \leqslant S < R$). Consider the subsets V_i of $V(G'')$ which may be associated, by the definition of G'', to each vertex v_i of G'. As each V_i has nR vertices, for each value of i there is at least one j such that the intersection set $U_j \cap V_i$ has more than n elements. For each value of i, denote one of these intersection sets (there is at least one but eventually more than one) by W_i. Each set W_i is entirely contained in one of the $R - S$ sets U_j. By the very definition of the chromatic number, $V(G')$ cannot be partitioned into less than R sets such that no edge exists linking one pair of vertices in at least one of the sets. Hence, in any distribution of the intersection sets W_i into the sets $U_1, \ldots, U_{R-S}$, there is at least one pair of intersection sets, say W_p and W_q, which are both contained in one of the $R - S$ sets, say U_j, and all vertices of W_p are linked to all vertices of W_q. They induce a graph whose minimum degree is greater than n. That is to say, the subgraph induced by U_j is not n-degenerate, since it contains a subgraph with minimum degree greater than n. This is a contradiction. Hence $\rho_n(G'') = R$.

Clearly, the graph G'' contains no triangles, hence no complete graphs K_p for $p \geqslant 3$. A graph G satisfying the requirements of the theorem may now be obtained. Take a complete graph K_d. As proved in [4], $\rho_n(K_d) = \{d/(n + 1)\}$ and the point-partition number of a disconnected graph equals the greatest point-partition number of its components. Hence the union of G'' and K_d is a graph G which satisfies the requirements of the theorem.

Acknowledgments

This research, done at the Mathematisches Institut der Technischen Universität München, was supported by a scholarship from Deutscher Akademischer Austauschdienst.

References

1. J. Mitchem, On graphs with prescribed clique number and point-arboricity, *J. London Math. Soc.* (2) **4** (1971), 333–336.
2. A. A. Zykov, "On some properties of linear complexes" (Russian), *Mat. Sbornik* **24** (1949), 163–188; *Amer. Math. Soc. Translation* (*Pamphlet series*) 79 (1952).

3. G. CHARTRAND, D. GELLER, AND S. HEDETNIEMI, Graphs with forbidden subgraphs, *J. Combinatorial Theory Ser. B* **10** (1971), 12–41.
4. D. R. LICK AND A. T. WHITE, k-degenerate graphs, *Canad. J. Math.* **22** (1970), 1082–1096.
5. F. HARARY, "Graph Theory," Addison-Wesley, Reading, Mass., 1969.